Newton's Law of Gravity Leads to Elliptical Orbits

According to ***Newton's Law of Gravity***, two force between two bodies of mass m_1 and m_2 is proportional to the inverse of the square of the distance between them, is attractive, and is exerted on the line between them. Put algebraically, if the two bodies are at positions $\vec{r}_1$ and $\vec{r}_2$, respectively, then the force on the first body is given by

$$m_1\vec{r}_1'' = -\frac{Gm_1m_2}{|\vec{r}_1-\vec{r}_2|^2}\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|} = -\frac{Gm_1m_2(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3} \quad (1)$$

while the force on the second body is given by an identical equation with the indices reversed,

$$m_2\vec{r}_2'' = -\frac{Gm_1m_2}{|\vec{r}_2-\vec{r}_1|^2}\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|} = -\frac{Gm_1m_2(\vec{r}_2-\vec{r}_1)}{|\vec{r}_1-\vec{r}_2|^3} \quad (2)$$

The number G is a constant, call ***Newton's Universal Gravitational Constant***. The second fraction in the center term of equation (1) represents a vector pointing from $\vec{r}_2$ to $\vec{r}_1$, while the corresponding term in equation (2) represents the same vector pointing in the opposite direction. Dividing the first equation by m_1 and the second equation by m_2 gives

$$\vec{r}_1'' = -\frac{Gm_2(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3} \quad (3)$$

$$\vec{r}_2'' = -\frac{Gm_1(\vec{r}_2-\vec{r}_1)}{|\vec{r}_1-\vec{r}_2|^3} \quad (4)$$

Subtracting equation (4) from equation (1),

$$\vec{r}_1''-\vec{r}_2'' = -\frac{Gm_2(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3} + \frac{Gm_1(\vec{r}_2-\vec{r}_1)}{|\vec{r}_1-\vec{r}_2|^3} = -\frac{G(m_2+m_1)(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3} \quad (5)$$

If we define a new vector

$$\vec{r} = \vec{r}_1-\vec{r}_2 \quad (6)$$

then equation (5) becomes

$$\vec{r}'' = -\frac{G(m_2+m_1)\vec{r}}{r^3} \quad (7)$$

It is typical to define the constant

$$\mu = G(m_1+m_2) \quad (8)$$

When we are dealing with an artificial satellite orbiting the earth, we can take

$$m_1 = m_{Earth} \approx 5.9742\times 10^{24}\,kg \quad (9)$$

$$m_2 = m_{satellite} \quad (10)$$

Typical satellite weights range from ≈100 *kg* (light) to ≈10,000 *kg* (heavy), so we can usually assume that

$$\mu = Gm_{earth} \approx 3.986 \times 10^{14} m^3 / sec^2 \tag{11}$$

The ***fundamental equation of motion*** is then

$$\vec{r}'' = -\mu\vec{r} / r^3 = -\mu\hat{r} / r^2 \tag{12}$$

where $\hat{r}$ is a unit vector in the direction $\vec{r}$.

Taking the cross product of (12) with $\vec{r}$ gives

$$\vec{r} \times \vec{r}'' = \vec{r} \times \left(\frac{-\mu\vec{r}}{r^3}\right) = -\frac{\mu}{r^3}\vec{r} \times \vec{r} = 0 \tag{13}$$

because the cross product of a vector with itself is zero. Furthermore, by the product rule,

$$\frac{d}{dt}(\vec{r} \times \vec{r}') = \vec{r} \times \vec{r}'' + \vec{r}' \times \vec{r}' = 0 + 0 = 0 \tag{14}$$

where the first zero follows from equation (13) and the second zero follows because the cross product of a vector with itself is zero. Whenever the derivative of a function is zero, then that function must be constant. Thus equation (14) tells us that $(\vec{r} \times \vec{r}')$ is a constant, i.e., it will not change as a function to time. Such a constant is called ***a constant of motion***; we will call this particular one $\vec{h}$,

$$\vec{h} = \vec{r} \times \vec{r}' \tag{15}$$

The constant $\vec{h}$ is called ***the angular momentum per unit mass***. Equation (15) is called the ***Law of Conservation of Angular Momentum***.

Taking the cross product of the fundamental equation of motion (12) with $\vec{h}$ gives

$$\vec{r}'' \times \vec{h} = -\mu\frac{\vec{r}}{r^3} \times (\vec{r} \times \vec{r}') = -\frac{\mu}{r^3}\vec{r} \times (\vec{r} \times \vec{r}') \tag{16}$$

Using the ***vector triple product*** identity,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{17}$$

equation (16) can be rewritten as

$$\vec{r}'' \times \vec{h} = -\frac{\mu}{r^3}\vec{r} \times (\vec{r} \times \vec{r}') = -\frac{\mu}{r^3}[(\vec{r} \cdot \vec{r}')\vec{r} - (\vec{r} \cdot \vec{r})\vec{r}'] \tag{18}$$

Since

$$\vec{r} \times \vec{r} = r^2 \tag{19}$$

this becomes

$$\vec{r}'' \times \vec{h} = -\frac{\mu}{r^3}\left[(\vec{r} \cdot \vec{r}')\vec{r} - r^2\vec{r}'\right] \tag{20}$$

By the quotient rule,

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{r\vec{r}' - \vec{r}r'}{r^2} = \frac{1}{r^3}\left[r^2\vec{r}' - r\vec{r}r'\right] \tag{21}$$

But

$$\hat{r} \cdot \vec{r}\,' = \text{component of } \frac{d}{dt}\vec{r} \text{ parallel to } \vec{r} = \frac{dr}{dt} = r' \tag{22}$$

Thus (multiply equation (22) through by *r*)

$$\vec{r} \cdot \vec{r}\,' = r\hat{r} \cdot \vec{r}\,' = rr' \tag{23}$$

Using (23) in (21) tells us that

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r^3}\left[r^2\vec{r}\,' - \vec{r}(\vec{r} \cdot \vec{r}\,')\right] \tag{24}$$

Comparing (20) with (24) gives

$$\vec{r}\,'' \times \vec{h} = \mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \tag{25}$$

Writing $\vec{r}\,'' = \frac{d}{dt}\vec{r}\,'$ equation (25) becomes

$$\left(\frac{d}{dt}\vec{r}\,'\right) \times \vec{h} = \mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \tag{26}$$

Since

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \tag{27}$$

for any two vectors, we can reverse the order of the cross product in (26)

$$\vec{h} \times \left(\frac{d}{dt}\vec{r}\,'\right) = -\mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \tag{28}$$

Multiplying both sides of the equation by dt and integrating,

$$\int \vec{h} \times \left(\frac{d}{dt}\vec{r}\,'\right) dt = -\int \mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) dt \tag{29}$$

Bringing the constants outside the integrals

$$\vec{h} \times \int \left(\frac{d}{dt}\vec{r}\,'\right) dt = -\mu \int \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) dt \tag{30}$$

Since *for any function f(t),*

$$\int \left(\frac{d}{dt} f(t)\right) dt = f(t) + C \tag{31}$$

or in terms of vectors,

$$\int \left(\frac{d}{dt} \vec{f}(t)\right) dt = \vec{f}(t) + \vec{C} \tag{32}$$

Therefore we can evaluate the integrals in (30),

$$-\vec{h} \times \vec{r}\,' = \mu \frac{\vec{r}}{r} + \vec{C} = \mu\left[\frac{\vec{r}}{r} + \frac{\vec{C}}{\mu}\right] = \mu\left[\frac{\vec{r}}{r} + \vec{e}\right] \tag{33}$$

where

$$\vec{e} = \vec{C} / \mu \tag{34}$$

is a constant of integration. Reversing the cross product in (33),

$$\vec{r}' \times \vec{h} = \mu\left[\frac{\vec{r}}{r} + \vec{e}\right] \tag{35}$$

Taking the dot product of (35) with $\vec{r}$ is

$$(\vec{r}' \times \vec{h}) \cdot \vec{r} = \mu\left[\frac{\vec{r}}{r} + \vec{e}\right] \cdot \vec{r} = \mu\left[\frac{\vec{r} \cdot \vec{r}}{r} + \vec{r} \cdot \vec{e}\right] = \mu(r + \vec{r} \cdot \vec{e}) \tag{36}$$

But from the definition of $\vec{h}$ (see equation 15),

$$(\vec{r}' \times \vec{h}) \cdot \vec{r} = (\vec{r} \times \vec{r}') \cdot \vec{h} = \vec{h} \cdot \vec{h} = h^2 \tag{37}$$

where the first equality follows from the vector identity $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{c} \times \vec{a}) \cdot b$. Using (37) in (36),

$$h^2 = \mu(r + \vec{r} \cdot \vec{e}) \tag{38}$$

Letting θ be the angle between $\vec{r}$ and $\vec{e}$, called the ***true anomaly***, we have

$$h^2 = \mu(r + \vec{r} \cdot \vec{e}) = \mu(r + re\cos\theta) = \mu r(1 + e\cos\theta) \tag{39}$$

Solving for *r*,

$$r = \frac{h^2 / \mu}{1 + e\cos\theta} \tag{40}$$

which is the equation of an ellipse if *e* is the eccentricity and the orbital semi-parameter is

$$p = h^2 / \mu \tag{41}$$

The fact that the orbit is an ellipse – which we have derived from Newton's laws of motion to arrive at equation (40) – is sometimes called ***Kepler's First Law of Motion***.

Since $p = a(1 - e^2)$ equation (41) is equivalent to

$$a(1 - e^2) = h^2 / \mu \tag{42}$$

or

$$h^2 = \mu a(1 - e^2) \tag{43}$$

Equation (43) is true at all points around the orbit; in particular, at perigee ($\theta = 0$) $\vec{r}$ is perpendicular to $\vec{r}'$, and therefore at perigee

$$\vec{h} = \vec{r} \times \vec{r}' = a(1 - e)V_{perigee} \tag{44}$$

where $V_{perigee}$ is the speed at perigee. Using (44) in (43),

$$a^2(1 - e)^2 V_{perigee}^2 = h^2 = \mu a(1 - e^2) \tag{45}$$

Solving for the speed at perigee,

$$V_{perigee}^2 = \frac{\mu}{a}\frac{1-e^2}{(1-e)^2} = \frac{\mu}{a}\frac{(1-e)(1+e)}{(1-e)^2} = \frac{\mu}{a}\left(\frac{1+e}{1-e}\right) \qquad (46)$$

Thus

$$V_{perigee} = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1+e}{1-e}} \qquad (47)$$

By a similar argument, the radius vector and velocity vector are also perpendicular at apogee (which occurs at $\theta = 180^\circ$, leading to

$$V_{apogee} = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1-e}{1+e}} \qquad (48)$$

When the eccentricity is zero, the orbit is precisely circular, and

$$V_{circular} = \sqrt{\frac{\mu}{a}} \qquad (49)$$