Review of Partial Derivatives

Definition of Partial Derivatives. Let $z = f(x,y)$ be a function. Then the partial derivative of f with respect to x is defined as

$$\frac{\partial f}{\partial x} \equiv f_x(x,y) \equiv \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

and the partial derivative of f with respect to y is defined as

$$\frac{\partial f}{\partial y} \equiv f_y(x,y) \equiv \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

Partial derivatives may be computed algebraically; all of the same rules that applied to regular derivatives also apply to partial derivatives. **The only trick to remember when taking a partial derivative is hold all other variables (besides the one we are differentiating with respect to) constant.**

Example. Let $f(x,y) = \frac{x^2}{1+y}$. Then

$$f_x(x,y) = \frac{\partial}{\partial x}\frac{x^2}{1+y} = \frac{1}{1+y}\frac{\partial}{\partial x}x^2 = \frac{2x}{1+y}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\frac{x^2}{1+y} = x^2\frac{\partial}{\partial y}\frac{1}{1+y} = x^2\frac{\partial}{\partial y}(1+y)^{-1} = \frac{-x^2}{(1+y)^2}$$

Example. Find the partial derivatives of $f(x,y) = y^2e^{3x}$

$$f_x(x,y) = \frac{\partial}{\partial x}\left(y^2e^{3x}\right) = y^2\frac{\partial}{\partial x}\left(e^{3x}\right) = y^2\cdot 3e^{3x} = 3y^2e^{3x}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(y^2e^{3x}\right) = e^{3x}\frac{\partial}{\partial y}\left(y^2\right) = e^{3x}\cdot 2y = 2ye^{3x}$$

Example. Find the partial derivative of $f(x,y,z) = \frac{x^2y^3}{z}$

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{x^2y^3}{z}\right) = \frac{y^3}{z}\frac{\partial}{\partial x}\left(x^2\right) = \frac{y^3}{z}\cdot 2x = \frac{2xy^3}{z}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{x^2y^3}{z}\right) = \frac{x^2}{z}\frac{\partial}{\partial y}\left(y^3\right) = \frac{x^2}{z}\cdot 3y^2 = \frac{3x^2y^2}{z}$$

$$f_z(x,y) = \frac{\partial}{\partial z}\left(\frac{x^2y^3}{z}\right) = x^2y^3\frac{\partial}{\partial z}\left(\frac{1}{z}\right) = x^2y^3\frac{\partial}{\partial z}\left(z^{-1}\right) = x^2y^3\left(-z^{-2}\right) = -\frac{x^2y^3}{z^2}$$

Second Order Partial Derivatives

We define higher order partial derivatives in much the same way as we did in single-variable calculus.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

With partial derivatives, we can also combine the variables, so there are more derivatives at each order. For example, we can differentiate f_x with respect to *y* and we can differentiate f_y with respect to *x*:

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

We can, of course, combine higher order derivatives in any order we like. For example:

$$f_{xxyx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^4 f}{\partial^2 x \partial y \partial x}$$

There are not as many partials as you might think, however, because of the following theorem:

The order of the partials can be reversed: $\boxed{f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}}$

Example. Find all the second-order partial derivatives of $f(x,y) = xy^2 + 3x^2e^y$ and show that $f_{xy} = f_{yx}$ by taking partials in both orders.

$$f_x = y^2 + 6xe^y \;\Rightarrow\; f_{xx} = 6e^y,\;\; f_{yx} = \frac{\partial}{\partial y}\left(y^2 + 6xe^y\right) = 2y + 6xe^y$$

$$f_y = 2xy + 3x^2e^y \;\Rightarrow\; f_{yy} = 2x + 3x^2e^y,\;\; f_{xy} = \frac{\partial}{\partial x}\left(2xy + 3x^2e^y\right) = 2y + 6xe^y = f_{yx}$$

Example. Repeat the above example for $f(x,y) = xe^y$

$$f_x = e^y \;\Rightarrow\; f_{xx} = 0,\quad f_{yx} = e^y$$

$$f_y = xe^y \;\Rightarrow\; f_{yy} = xe^y,\quad f_{xy} = e^y = f_{yx}$$

Chain Rule

If $z(t) = f\left(x(t),\ y(t)\right)$ then the chain rule is

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

In general, if z is a function of any number of variables $x(t),\ y(t),\ z(t),\ w(t),\ \dots$, each of which can be expressed as a function of only t (and not of any other parameter),

$$\frac{d}{dt}f\left(x,y,z,w,\dots\right) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial w}\frac{dw}{dt} + \dots$$

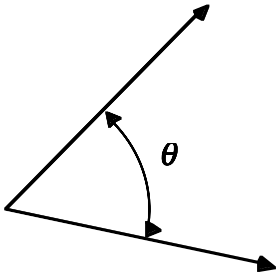
Example. Suppose $f(x,y) = x\sin y$, where $x = t^2$ and $y = 2t+1$. Then by the chain rule we have

$$\begin{aligned} f'(t) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial}{\partial x}\left(x\sin y\right)\frac{d}{dt}\left(t^2\right) + \frac{\partial}{\partial y}\left(x\sin y\right)\frac{d}{dt}(2t+1) \\ &= \left(\sin y\right)(2t) + \left(x\cos y\right)(2) \\ &= 2t\sin(2t+1) + 2t^2\cos(2t+1) \end{aligned}$$

Vector Products

There are two types of products between vectors, one of which produces a vector and the other produces a scalar

- The dot product $\vec{v}\cdot\vec{w}\longrightarrow scalar$
- The cross product $\vec{v}\times\vec{w}\longrightarrow vector$



The Dot Product is defined geometrically

$$\vec{v}\cdot\vec{w}=|v||w|\cos\theta$$

where q is the angle between the two vectors as shown in the figure. Algebraically, if

$$\vec{v}=\vec{i}\,v_1+\vec{j}\,v_2+\vec{k}\,v_3 \quad \text{and} \quad \vec{w}=\vec{i}\,w_1+\vec{j}\,w_2+\vec{k}\,w_3$$

Then $\vec{v}\cdot\vec{w}=\vec{w}\cdot\vec{v}=v_1w_1+v_2w_2+v_3w_3$

Example. Suppose that $\vec{u}=3\vec{i}+4\vec{j}+5\vec{k}$ and $\vec{v}=7\vec{i}+8\vec{j}+9\vec{k}$

Then $\vec{u}\cdot\vec{v}=(3)(7)+(4)(8)+(5)(9)=21+32+45=98$

Properties of the dot product

1. $\vec{v}\cdot\vec{w}=\vec{w}\cdot\vec{v}$
2. $\vec{v}\cdot(a\vec{w})=(a\vec{v})\cdot\vec{w}=a(\vec{v}\cdot\vec{w})$
3. $(\vec{v}+\vec{u})\cdot\vec{w}=\vec{v}\cdot\vec{w}+\vec{u}\cdot\vec{w}$
4. $\vec{v}$ and $\vec{w}$ are perpendicular only if $\vec{v}\cdot\vec{w}=0$.

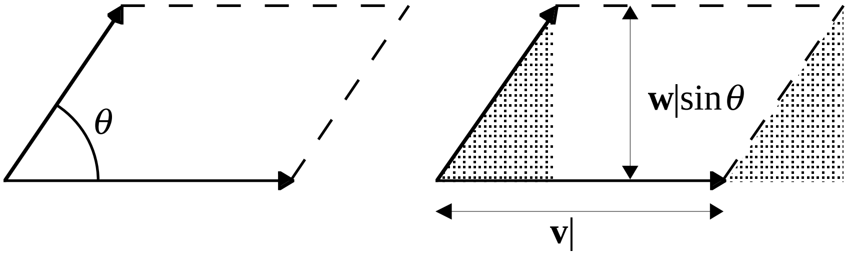
The Cross Product

The cross product is a product between vectors that results in a vector. It is defined as a vector with the following properties:

- Its length is equal to $|\vec{v}\times\vec{w}|=|\vec{v}||\vec{w}|\sin\theta$
- direction is perpendicular to the plane that contains v and w
- Its orientation (up vs. down) is according to the right hand rule

Right-Hand Rule: Place $\vec{u}$ and $\vec{v}$ so that their tails coincide and curl the fingers of your right hand from through the angle from $\vec{u}$ to $\vec{v}$. Your thumb is pointing in the direction of $\vec{u}\times\vec{v}$

The cross product gives the area of the parallelogram formed by the two vectors:



We can also calculate the cross product algebraically from the components of the individual vectors if we use determinants.

$$\vec{v}\times\vec{w}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}=\vec{i}\begin{vmatrix}v_2 & v_3\\ w_2 & w_3\end{vmatrix}-\vec{j}\begin{vmatrix}v_1 & v_3\\ w_1 & w_3\end{vmatrix}+\vec{k}\begin{vmatrix}v_1 & v_3\\ w_1 & w_3\end{vmatrix}$$

$$=\vec{i}\left(v_2w_3-v_3w_2\right)-\vec{j}\left(v_1w_3-v_3w_1\right)+\vec{k}\left(v_1w_2-v_2w_1\right)$$

Determinant of a Matrix

$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}\equiv\begin{vmatrix}a & b\\ c & d\end{vmatrix}\equiv ad-bc$$

$$\det\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}=\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}=a\begin{vmatrix}e & f\\ h & i\end{vmatrix}-b\begin{vmatrix}d & f\\ g & i\end{vmatrix}+c\begin{vmatrix}d & e\\ g & h\end{vmatrix}$$

$$=a(ei-fh)-b(di-fg)+c(dh-eg)$$

Properties of the Cross Product

(1) $\vec{w}\times\vec{v}=-\vec{v}\times\vec{w}$

(2) $\left(a\vec{v}\right)\times\vec{w}=a\left(\vec{v}\times\vec{w}\right)=\vec{v}\times\left(a\vec{w}\right)$

(3) $\vec{u}\times\left(\vec{v}+\vec{w}\right)=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}$

(4) $\vec{u}\times\vec{v}=\mathbf{0}$ if and only if the vectors are parallel (assuming that $\vec{u},\vec{v}\neq\vec{0}$)

(5) $\vec{i}\times\vec{j}=\vec{k},\ \vec{j}\times\vec{k}=\vec{i},\ \vec{k}\times\vec{i}=\vec{j}$ (cyclic cross products)

(6) $\vec{j}\times\vec{i}=-\vec{k},\ \vec{k}\times\vec{j}=-\vec{i},\ \vec{i}\times k=-\vec{j}$ (acyclic cross products)

(7) $\vec{v}\times\vec{v}=0$

Vector Operators: Gradient, Divergence and Curl

The **gradient of a function of three variables** is the vector

$$\text{grad}\, f\equiv\nabla f(x,y,z)\equiv\vec{i}\,\frac{\partial f}{\partial x}+\vec{j}\,\frac{\partial f}{\partial y}+\vec{k}\,\frac{\partial f}{\partial z}$$

We can think of the gradient symbol ∇ as a operator that behaves exactly like a vector with the exception of the fact that it operates on the single item immediately to its right.

In this sense, we can think of ∇ as vector version of $\frac{\partial}{\partial x}$.

Specifically, ∇ represents the “vector” $\vec{\nabla}=\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}$

Example. $f = xe^{y^2+z}$

Then

$$\begin{aligned}\vec{\nabla} f &= \vec{\nabla}\left(xe^{y^2+z} \right) = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right)\left(xe^{y^2+z} \right) \\ &= \vec{i}\,\frac{\partial}{\partial x}\left(xe^{y^2+z} \right) + \vec{j}\,\frac{\partial}{\partial y}\left(xe^{y^2+z} \right) + \vec{k}\,\frac{\partial}{\partial z}\left(xe^{y^2+z} \right) \\ &= \vec{i}\,e^{y^2+z} + \vec{j}\,2xye^{y^2+z} + \vec{k}xe^{y^2+z}\end{aligned}$$

Since we are treating ∇ "like a vector," we can subject to all of the usual vector operations, such as dot product and cross product. We will define the following operations using ∇

Operation	Name of Operator	Input	Output
∇	gradient	scalar	vector
$\nabla \cdot$	divergence (dot product) read as "del dot ..."	vector	scalar
$\nabla \times$	curl (cross product) read as "del cross ..."	vector	vector
$\nabla \cdot \nabla$ or ∇^2	Laplacian read as "del squared ..."	scalar	scalar

Both the divergence and curl operate on vector functions. A vector function is a function

$$\vec{F}(x,y,z) = \vec{i}\,F_1(x,y,z) + \vec{j}F_2(x,y,z) + \vec{k}F_3(x,y,z)$$

where the functions $F_1,\ F_2,\ F_3$ are all real valued functions.

Example of a Vector Function. $\vec{F}(x,y,z) = x\vec{i} + \left(y^2x + z\right)\vec{j} + e^x\vec{k}$

The Divergence Operator is defined as "del dot a vector function",

$$\begin{aligned}\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} &= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) \cdot \left(\vec{i}\,F_1 + \vec{j}F_2 + \vec{k}F_3 \right) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\end{aligned}$$

Example. Find the divergence of $\vec{F}(x,y,z) = x\vec{i} + \left(y^2x + z\right)\vec{j} + e^x\vec{k}$

$$\begin{aligned}\vec{\nabla} \cdot \vec{F} &= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) \cdot \left(x\vec{i} + \left(y^2x + z\right)\vec{j} + e^x\vec{k} \right) \\ &= \frac{\partial x}{\partial x} + \frac{\partial}{\partial y}\left(y^2x + z\right) + \frac{\partial}{\partial z}e^x = 1 + 2xy + 0 = 1 + 2xy\end{aligned}$$

The Curl Operator is defined as “del cross f”

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) \times \left(\vec{i}\,F_1 + \vec{j}F_2 + \vec{k}F_3 \right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \vec{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Example. Suppose $\vec{F} = e^{y^2}\vec{i} \;+\; 2xye^{y^2}\vec{j} \;+\; \vec{k}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{y^2} & 2xye^{y^2} & 1 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial(1)}{\partial y} - \frac{\partial}{\partial z}[2xye^{y^2}] \right) + \vec{j}\left(\frac{\partial}{\partial z}e^{y^2} - \frac{\partial(1)}{\partial x} \right) + \vec{k}\left(\frac{\partial}{\partial x}[2xye^{y^2}] - \frac{\partial e^{y^2}}{\partial y} \right)$$

$$= \vec{i}\,(0-0) + \vec{j}\,(0-0) + \vec{k}\left(2ye^{y^2} - 2ye^{y^2} \right) = \mathbf{0}$$

The Laplacian Operator is the dot product of ∇ with itself, or the divergence of the gradient. It is sometimes read as “del squared”.

$$\nabla^2 f(x,y,z) = \vec{\nabla} \cdot \left(\vec{\nabla} f(x,y,z) \right)$$

$$= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) \cdot \left[\left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) f \right]$$

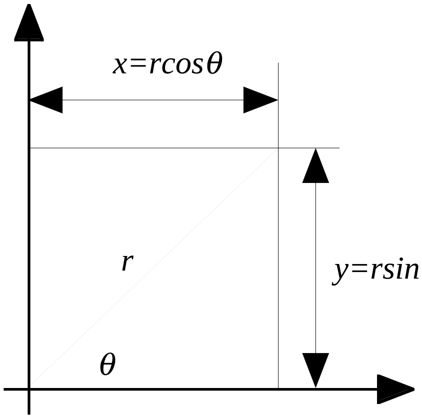
$$= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) \cdot \left(\vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian operates on a scalar and its output is a scalar.

Polar Coordinates

In polar coordinates, instead of using the distances x and y from the origin to locate a point, we use a single distance r and an orientation angle with respect to the x-axis.

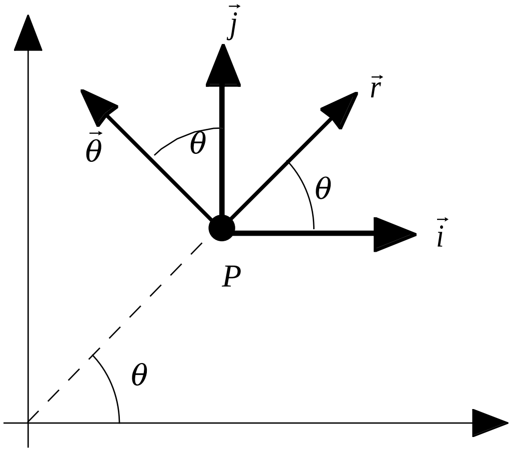


The two coordinate systems are related to each other as follows:

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \theta = \arctan\frac{y}{x}$$

We can also define unit vectors $\vec{r}$ and $\vec{\theta}$ at any given point in space; they can be related to the cartesian unit vectors $\vec{i}$ and $\vec{j}$ from the following geometry.



Since all the vectors shown are unit vectors, we observe that:

x component of $\hat{r}$ is $\cos\theta$; x component of $\hat{\theta}$ is $-\sin\theta$

y component of $\hat{r}$ is $\sin\theta$; y component of $\hat{\theta}$ is $\cos\theta$

Therefore

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$
$$\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

Exercise: Find expressions for $\hat{i}$ and $\hat{j}$ in terms of $\hat{r}$ and $\hat{\theta}$.

Solution. Multiply the first expression by $\sin\theta$ and the second expression by $\cos\theta$

$$\hat{r}\sin\theta = \sin\theta\cos\theta\hat{i} + \sin^2\theta\hat{j}$$
$$\hat{\theta}\cos\theta = -\cos\theta\sin\theta\hat{i} + \cos^2\theta\hat{j}$$

Adding the two expressions,

$$\hat{r}\sin\theta + \hat{\theta}\cos\theta = \sin^2\theta\hat{j} + \cos^2\theta\hat{j} = \hat{j}$$

which gives an expression for $\hat{j}$ in terms of $\hat{r}$ and $\hat{\theta}$.

To get a similar expressionf or $\hat{i}$, multiply the expression for $\hat{r}$ by $\cos\theta$ and the expression for $\hat{\theta}$ by $\sin\theta$,

$$\hat{r}\cos\theta = \cos^2\theta\hat{i} + \cos\theta\sin\theta\hat{j}$$
$$\hat{\theta}\sin\theta = -\sin^2\theta\hat{i} + \cos\theta\sin\theta\hat{j}$$

Subtract the bottom expression from the top,

$$\hat{r}\cos\theta - \hat{\theta}\sin\theta = \cos^2\theta\hat{i} + \sin^2\theta\hat{i} = \hat{i}$$

Summarizing the conversion expressions, we have

$$\begin{aligned} i &= \hat{r}\cos\theta - \hat{\theta}\sin\theta \\ \hat{j} &= \hat{r}\sin\theta + \hat{\theta}\cos\theta \end{aligned} \quad \text{and} \quad \begin{aligned} \vec{r} &= \cos\theta\vec{i} + \sin\theta\vec{j} \\ \vec{\theta} &= -\sin\theta\vec{i} + \cos\theta\vec{j} \end{aligned}$$

Example. Find an expression for the vector field $\vec{F} = 3x\hat{i} + 4x^2y\hat{j}$ in polar coordinates.

Solution.

$$\begin{aligned} \vec{F} &= 3x\hat{i} + 4x^2y\hat{j} \\ &= 3(r\cos\theta)(\hat{r}\cos\theta - \hat{\theta}\sin\theta) + 4(r\cos\theta)^2(r\sin\theta)(\hat{r}\cos\theta + \hat{\theta}\sin\theta) \\ &= 3r\cos^2\theta\hat{r} - 3r\cos\theta\sin\theta\hat{\theta} + 4r^3\cos^3\theta\sin\theta\hat{r} + 4r^3\cos^2\theta\sin^2\theta\hat{\theta} \\ &= (3r\cos^2\theta + 4r^3\cos^3\theta\sin\theta)\hat{r} + (4r^3\cos^2\theta\sin^2\theta - 3r\cos\theta\sin\theta)\hat{\theta} \\ &= r\cos^2\theta(3 + 4r^2\cos\theta\sin\theta)\hat{r} + r\cos\theta\sin\theta(4r^2\cos\theta\sin\theta - 3)\hat{\theta} \end{aligned}$$

Exercise: Recall that the gradient in two dimensions in cartesian coordinates is $\nabla f(x,y) = \hat{i}\dfrac{\partial f}{\partial x} + \hat{j}\dfrac{\partial f}{\partial y}$. Find an expression for the gradient in polar coordinates, i .e., find the functions *g* and *h* so that $\nabla f(r,\theta) = \hat{r}g(r,\theta) + \hat{\theta}h(r,\theta)$. Hint: you must use the chain rule and the expressions for transformation of coordinates.