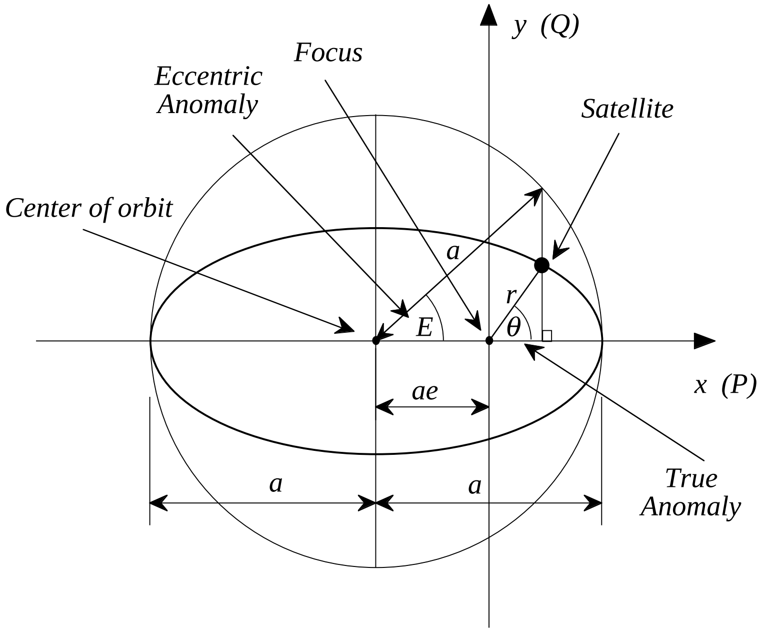
Motion in the plane of the orbit

The motion of a satellite moving in an elliptical orbit is illustrated in the following figure. By definition, the motion is counter-clockwise. The x-axis and y-axis in the figure correspond to the $\vec{P}$ and $\vec{Q}$ axes in the PQW frame



Two special angles are defined, θ, the ***true anomaly***, and E, the ***eccentric anomaly***, as illustrated in the figure. The x and y coordinates of the satellite in this reference frame are

$$x = r\cos\theta \qquad y = r\sin\theta \tag{1}$$

In terms of the eccentric anomaly,

$$a\cos E = ae + x \tag{2}$$

Solving for x,

$$x = a(\cos E - e) \tag{3}$$

From the equation of an ellipse, we also know that

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{4}$$

Using the expression for $\cos\theta = x / r$ from equation (1) in equation (4),

$$r = \frac{a(1-e^2)}{1+ex/r} \tag{5}$$

Cross-multiplying leads to

$$r + ex = a(1-e^2) \tag{6}$$

Solving for r,

$$r = a(1-e^2) - ex \tag{7}$$

Substituting equation (3) into equation (7)

$$
\begin{aligned}
r &= a(1-e^2) - ea(\cos E - e) \\
&= a - ae^2 - ae\cos E + ae^2 \\
&= a - ae\cos E \\
&= a(1-e\cos E)
\end{aligned} \tag{8}
$$

By the Pythagorean theorem,

$$r^2 = x^2 + y^2 \tag{9}$$

so that we can solve for y to get

$$y = \sqrt{r^2 - x^2} \tag{10}$$

Using the last line of equation (8) and equation (3) in equation (10) and simplifying,

$$
\begin{aligned}
y &= \sqrt{a^2(1-e\cos E)^2 - a^2(\cos E - e)^2} \\
&= a\sqrt{(1-e\cos E)^2 - (\cos E - e)^2} \\
&= a\sqrt{1 - 2e\cos E + e^2\cos^2 E - \cos^2 E + 2e\cos E - e^2} \\
&= a\sqrt{1 + e^2\cos^2 E - \cos^2 E - e^2} \\
&= a\sqrt{1 - \cos^2 E + e^2(\cos^2 E - 1)} \\
&= a\sqrt{(1-e^2)(1-\cos^2 E)} \\
&= a\sqrt{1-e^2}\sin E
\end{aligned} \tag{11}
$$

because, in the last line, $1-\cos^2 E = \sin^2 E$.

Summarizing, the coordinates of the satellite in terms of both the true and the eccentric anomaly are

$$
\begin{aligned}
r &= a(1-e\cos E) \\
x &= r\cos\theta = a(\cos E - e) \\
y &= r\sin\theta = a(1-e^2)^{1/2}\sin E
\end{aligned} \tag{12}
$$

Differentiating with respect to time,

$$\frac{dx}{dt} = \frac{d}{dt}\left[a(\cos E - e)\right] = -a\sin E\frac{dE}{dt} \tag{13}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[a(1-e^2)^{1/2}\sin E\right] = a(1-e^2)^{1/2}\cos E\frac{dE}{dt} \tag{14}$$

$$\frac{dr}{dt} = \frac{d}{dt}\left[a(1-e\cos E)\right] = ae\sin E\frac{dE}{dt} \tag{15}$$

From the defiition of angular momentum,

$$\vec{h} = \vec{r} \times \frac{d\vec{r}}{dt} = \begin{pmatrix} \vec{P} & \vec{Q} & \vec{W} \\ a(\cos E - e) & a(1-e^2)^{1/2}\sin E & 0 \\ -a\sin E\dfrac{dE}{dt} & a(1-e^2)^{1/2}\cos E\dfrac{dE}{dt} & 0 \end{pmatrix} \tag{16}$$

Expanding out the cross product,

$$\begin{aligned}\vec{h} &= \left[a^2(\cos E - e)(1-e^2)^{1/2}\cos E\frac{dE}{dt} + a^2(1-e^2)^{1/2}\sin^2 E\frac{dE}{dt}\right]\vec{W}\\ &= a^2(1-e^2)^{1/2}\frac{dE}{dt}\left[(\cos E - e)\cos E + \sin^2 E\right]\vec{W}\\ &= a^2(1-e^2)^{1/2}\frac{dE}{dt}\left[\cos^2 E - e\cos E + \sin^2 E\right]\vec{W}\\ &= a^2(1-e^2)^{1/2}\frac{dE}{dt}[1-e\cos E]\vec{W}\end{aligned} \tag{17}$$

Thus

$$h^2 = \vec{h}\cdot\vec{h} = a^4(1-e^2)\left(\frac{dE}{dt}\right)^2(1-e\cos E)^2 \tag{18}$$

In an earlier lecture we found that

$$a(1-e^2) = p = h^2/\mu \tag{19}$$

or equivalently,

$$h^2 = \mu a(1-e^2) \tag{20}$$

Equating the two expressions for h^2 (equations 18 and 20),

$$\mu a(1-e^2) = a^4(1-e^2)\left(\frac{dE}{dt}\right)^2(1-e\cos E)^2 \tag{21}$$

Canceling common factors

$$\mu = a^3\left(\frac{dE}{dt}\right)^2(1-e\cos E)^2 \tag{22}$$

Dividing by a^3 and taking the (positive) square root,

$$\frac{\sqrt{\mu}}{a^{3/2}} = (1-e\cos E)\frac{dE}{dt} \tag{23}$$

If we multiply across by dt and integrate, from the time of perigee (time $T_{Perigee}$), to some later time T, at which point the eccentric anomaly is assumed to be E_T, we have

$$\frac{\sqrt{\mu}}{a^{3/2}}\int_{T_{perigee}}^{T} dt = \int_0^{E_T}(1-e\cos E)dE \tag{24}$$

Evaluating the integrals,

$$\frac{\sqrt{\mu}}{a^{3/2}}(T - T_{Perigee}) = (E - e\sin E)\Big|_0^{E_T} = E_T - e\sin E_T \tag{25}$$

Using the usual notation of a lower case t for time and $E = E(t)$ be the eccentric anomaly,

$$n(t - T_P) = E - e\sin E \tag{26}$$

where

$$n = \frac{\sqrt{\mu}}{a^{3/2}} \tag{27}$$

is called the ***mean motion***.. Rearranging equation (26),

$$t - T_P = \frac{E - e\sin E}{n} \tag{28}$$

If we let τ be the ***period*** (or more technically, the ***anomalistic period***) of the orbit, i.e., the time it takes for the satellite to go from one perigee to the next, then in equation (28), $\tau = t - T_P$ and $E = 2\pi$, so that

$$\tau = \frac{2\pi - e\sin 2\pi}{n} = \frac{2\pi}{n} \tag{29}$$

Solving (29) for $n = 2\pi / \tau$ and substituting it into (27),

$$\frac{2\pi}{\tau} = n = \frac{\sqrt{\mu}}{a^{3/2}} \tag{30}$$

Cross multiplying,

$$2\pi a^{3/2} = \tau\sqrt{\mu} \tag{31}$$

Squaring equation (31),

$$4\pi^2 a^3 = \tau^2 \mu \tag{33}$$

which is sometimes call ***Kepler's Third Law of Motion***: that the square of the period is proportional to the cube of the semi-major axis.

Equation 26 is sometimes used to ***define*** a third angle, *M*, the ***mean anomaly***, as

$$M = n(t - T_P) = E - e\sin E \tag{34}$$

It is important to remember that the Mean Anomaly while it has units of radians, has no geometrical meaning in terms of the picture of the elliptical orbit. What it does have is a physical meaning, given by the first equality of equation (34), that by definition,

$$M = n(t - T_P) \tag{35}$$

The mean anomaly is the equivalent angle that a satellite with the same semi-major axis, but moving in a circle, would move through, in the same amount of time that the satellite moves through an eccentric anomaly *E* starting at perigee. This construction is very useful because *M* increases linearly with time. Dropping the middle of equation (34) we have ***Kepler's Equation***,

$$M = E - e\sin E \tag{36}$$

Using equations (35) and (36) it is possible to predict the position of the satellite at any time *t* as follows:

(1) Use (35) to compute *M*
(2) Given *M*, solve (36) numerically for *E*.
(3) Given *E*, use equations (12) to determine *x* and *y* in the PQW frame
(4) Use a coordinate transformation to convert back to the XYZ frame

Step (2) is the only difficult part because equation (36) can not be solved analytically for *E*. The most common method is to use Newton's Method to find the root of

$$f(E) = E - e\sin E - M \tag{37}$$

According to Newton's method, we first make an educated guess, call it E_1, and then proceed to compute better guesses as

$$E_n = E_{n-1} - \frac{f(E_{n-1})}{f'(E_{n-1})} \tag{38}$$

Differentiating (37), this means

$$E_n = E_{n-1} - \frac{E_{n-1} - e\sin E_{n-1} - M}{1 - e\cos E_{n-1}} \tag{39}$$

A relatively good first guess is to use

$$E_1 = M \tag{40}$$