Discrete Mathematics with Applications, 3rd Edition
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Tips for Success with Proofs and Disproofs

Make sure your proofs are genuinely convincing. Express yourself carefully and completely – but concisely! Write in complete sentences, but don't use an unnecessary number of words.

Disproof by Counterexample

- To disprove a universal statement, give a counterexample.
- Write the word "Counterexample" at the beginning of a counterexample.
- Write counterexamples in complete sentences.
- Give values of the variables that you believe show the property is false.
- Include the computations that prove beyond any doubt that these values really do make the property false.

All Proofs

- Write the word "Proof" at the beginning of a proof.
- Write proofs in complete sentences.
- Start each sentence with a capital letter and finish with a period.

Direct Proof

- Begin each direct proof with the word "Suppose."
- In the "Suppose" sentence:
  - Introduce a variable or variables (indicating the general set they belong to - e.g., integers, real numbers etc.), and
  - Include the hypothesis that the variables satisfy.
- Identify the conclusion that you will need to show in order to complete the proof.
- Reason carefully from the "suppose" to the "conclusion to be shown."
- Include the little words (like "Then," "Thus," "So," "It follows that") that make your reasoning clear.
- Give a reason to support each assertion you make in your proof.

Proof by Contradiction

- Begin each proof by contradiction by writing "Suppose not. That is, suppose...," and continue this sentence by carefully writing the negation of the statement to be proved.
- After you have written the "suppose," you need to show that this supposition leads logically to a contradiction.
- Once you have derived a contradiction, you can conclude that the think you supposed is false. Since you supposed that the given statement was false, you now know that the given statement is true.

Proof by Contraposition

- Look to see if the statement to be proved is a universal conditional statement.
- If so, you can prove it by writing a direct proof of its contrapositive.

Formats for Proving Formulas by Mathematical Induction

When using mathematical induction to prove a formula, students are sometimes tempted to present their proofs in a way that assumes what is to be proved. There are several formats you can use, besides the one shown most frequently in the textbook, to avoid this fallacy. A crucial point is this:

> If you are hoping to prove that an equation is true but you haven't yet done so, either preface it with the words "We must show that" or put a question mark above the equal sign.

Format 1 (the format used most often in the textbook for the inductive step): Start with the left-hand side (LHS) of the equation to be proved and successively transform it using definitions, known facts from basic algebra, and (for the inductive step) the inductive hypothesis until you obtain the right-hand side (RHS) of the equation.

Format 2 (the format used most often in the textbook for the basis step): Transform the LHS and the RHS of the equation to be proved *independently*, one after the other, until both sides are shown to equal the same expression. Because two quantities equal to the same quantity are equal to each other, you can conclude that the two sides of the equation are equal to each other.

Format 3: This format is just like Format 2 except that the computations are done in parallel. But in order to avoid the fallacy of assuming what is to be proved, do NOT put an equal sign between the two sides of the equation until the very last step. Separate the two sides of the equation with a vertical line.

Format 4: This format is just like Format 3 except that the two sides of the equation are separated by an equal sign with a question mark on top: $\stackrel{?}{=}$

Format 5: Start by writing something like "We must show that" and the equation you want to prove true. In successive steps, indicate that this equation is true if, and only if, ($\Leftrightarrow$) various other equations are true. But be sure that both the directions of your "if and only if" claims are correct. In other words, be sure that the $\Leftarrow$ direction is just as true as the $\Rightarrow$ direction. If you finally get down to an equation that is known to be true, then because each subsequent equation is true *if, and only if*, the previous equation is true, you will have shown that the original equation is true.

Example: Prove that for each integer $n \geq 1$,

$$\boxed{1 + 3 + 5 + \cdots + (2n - 1) = n^2} \leftarrow \text{This is the equation.}$$

Proof that the equation is true for $n = 1$:

Solution (Format 2):
When $n = 1$, the LHS of the equation equals 1, and the RHS equals 1^2 which also equals 1. So the equation is true for $n = 1$.

Solution (Format 5):
When $n = 2$, we must show that $1 = 1^2$. Because this is true, the equation is true for $n = 1$.

Proof that if the equation is true for $n = k$ then it is true for $n = k+1$:

Solution (Format 2):
Suppose that for some integer $k \geq 1$, $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis.]*
We must show that $1+3+5+\cdots+(2k+1) = (k+1)^2$.
But the LHS of the equation to be shown is

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &= 1+3+5+\cdots+(2k-1)+(2k+1) && \text{by making the next-to-last term explicit} \\ &= k^2+(2k+1) && \text{by inductive hypothesis.} \end{aligned}$$

And the RHS of the equation to be shown is

$$(k+1)^2 = k^2+2k+1 \qquad \text{by basic algebra.}$$

So the LHS and the RHS are equal to the same quantity, and thus they are equal to each other *[as was to be shown].*

Solution (Format 3):
Suppose that for some integer $k \geq 1$, $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis.]*
We must show that $1+3+5+\cdots+(2k+1) = (k+1)^2$.
But

$1+3+5+\cdots+(2k+1)$	$(k+1)^2$
$= 1+3+5+\cdots+(2k-1)+(2k+1)$	
by making the next-to-last term explicit	
$= k^2+(2k+1)$	
by inductive hypothesis	
$= k^2+2k+1$	$= k^2+2k+1$
by basic algebra	by basic algebra

So the LHS and the RHS are equal to the same quantity, and thus they are equal to each other *[as was to be shown].*

Solution (Format 4):
Suppose that for some integer $k \geq 1$, $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis.]*
We must show that $1+3+5+\cdots+(2k+1) = (k+1)^2$.
But

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &\stackrel{?}{=} (k+1)^2 \\ 1+3+5+\cdots+(2k-1)+(2k+1) &\stackrel{?}{=} k^2+2k+1 \\ \text{by making the next-to-last term explicit} & \qquad \text{by basic algebra} \\ k^2+(2k+1) &\stackrel{?}{=} k^2+2k+1 \\ \text{by inductive hypothesis} & \\ k^2+2k+1 &= k^2+2k+1 \\ \text{by basic algebra} & \end{aligned}$$

So the LHS and the RHS are equal to the same quantity, and thus they are equal to each other *[as was to be shown].*

Solution (Format 5):
Suppose that for some integer $k \geq 1$, $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis.]*
We must show that $1+3+5+\cdots+(2k+1) = (k+1)^2$.
But this equation is true if, and only if, ($\Leftrightarrow$)

$$\begin{aligned} & 1+3+5+\cdots+(2k-1)+(2k+1) = (k+1)^2 && \text{by making the next-to-last term explicit} \\ \Leftrightarrow\ & k^2+(2k+1) = (k+1)^2 && \text{by inductive hypothesis} \\ \Leftrightarrow\ & k^2+2k+1 = (k+1)^2 && \end{aligned}$$

which is true by basic algebra. Thus the equation to be shown is also true.