9.3. Improper Integrals: limits at ±∞

An improper integral with limits at infinity is any integral in which one (or both) of the limits is (are) either ±∞.

Example 1. $\int_0^{\infty} \frac{\sin x}{x} dx,\ \int_{-\infty}^{\infty} e^{-x^2/2\pi} dx,\ \int_{-\infty}^{23} e^x dx$ are all improper integrals with infinite limits.

DEFINITION. IMPROPER INTEGRALS WITH ONE INFINITE LIMIT. If the following limits exist, the integrals are said to **CONVERGE**.

$$\int_a^{\infty} f(x)dx \equiv \lim_{b\to\infty} \int_a^b f(x)dx$$

$$\int_{-\infty}^{b} f(x)dx \equiv \lim_{a\to-\infty} \int_a^b f(x)dx$$

If the limits do not exist, the integrals are said to **DIVERGE**.

IMPROPER INTEGRALS WITH BOTH LIMITS INFINITE:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

Example 2. $\int_0^{\infty} e^{-x}dx = \lim_{b\to\infty} \int_0^b e^{-x}dx = \lim_{b\to\infty} (-e^{-x})\Big|_0^b = \lim_{b\to\infty} (-e^{-b}+1) = 1$

Example 3. Find $\int_1^{\infty} \frac{\ln x}{x} dx$

Let $u = \ln x$; then $du = dx/x$; $x = 1 \Rightarrow u = \ln 1 = 0$; $x = \infty \Rightarrow u = \ln\infty = \infty$

$$\int_1^{\infty} \frac{\ln x}{x} dx = \int_0^{\infty} u du = \lim_{b\to\infty} \int_0^b u du = \lim_{b\to\infty} \frac{u^2}{2}\Big|_0^b = \lim_{b\to\infty} \frac{b^2}{2} = \infty \text{ (DIVERGES)}$$

Example 4. Find $\int_1^{\infty} xe^{-x^2} dx$

Let $u = -x^2$; $du = -2xdx$; $x = 1 \Rightarrow u = -1$; $x = \infty \Rightarrow u = -\infty$

$$\int_1^{\infty} xe^{-x^2} dx = \int_{-1}^{-\infty} e^u(-du/2) = -\frac{1}{2}\lim_{b\to-\infty} \int_{-1}^{b} e^u du = -\frac{1}{2}\lim_{b\to-\infty} e^u\Big|_{-1}^{b}$$

$$= -\frac{1}{2}\lim_{b\to-\infty} (e^b - e^{-1}) = \frac{1}{2e}$$

Example 5. Find $\int_e^{\infty} \frac{1}{x\ln x} dx$

Let $u = \ln x$; then $du = dx/x$; $x = e \Rightarrow u = \ln e = 1$; $x = \infty \Rightarrow u = \ln\infty = \infty$

$$\int_e^{\infty} \frac{1}{x\ln x} dx = \int_1^{\infty} \frac{1}{u} du = \lim_{b\to\infty} \ln u\Big|_1^b = \ln\infty = \infty$$

Example 6. Find $\int_e^{\infty} \frac{\ln x}{x^2} dx$

Let $u = \ln x \Rightarrow x = e^u$; then $du = dx / x$; $x = e \Rightarrow u = \ln e = 1$; $x = \infty \Rightarrow u = \ln \infty = \infty$ (Same as previous example). Then

$$\int_e^{\infty} \frac{\ln x}{x^2} dx = \int_e^{\infty} (\ln x) \frac{1}{x} \frac{dx}{x} = \int_1^{\infty} \frac{u}{e^u} du = \lim_{b \to \infty} \int_1^b u e^{-u} du = \lim_{b \to \infty} \int_1^b x e^{-x} dx$$

Integrate by parts. Let $u = x$, $dv = e^{-x} dx$; then $du = dx$; $v = -e^{-x}$

$$\begin{aligned}
\lim_{b \to \infty} \int_1^b x e^{-x} dx &= \lim_{b \to \infty} \left((x)(-e^{-x}) \Big|_1^b - \int_1^b (-e^{-x}) dx \right) \\
&= \lim_{b \to \infty} \left(-b e^{-b} + \frac{1}{e} + \int_1^b e^{-x} dx \right) \\
&= \lim_{b \to \infty} \left(-b e^{-b} + \frac{1}{e} + (-e^{-x}) \Big|_1^b \right) \\
&= \lim_{b \to \infty} \left(-b e^{-b} + \frac{1}{e} + -e^{-b} + \frac{1}{e} \right) \\
&= \lim_{b \to \infty} \left(-(1+b) e^{-b} + \frac{2}{e} \right) \\
&= \frac{2}{3} - \lim_{b \to \infty} \frac{b+1}{e^b}
\end{aligned}$$

The remaining limit is ∞/∞ so we can use L'Hôpital's rule; the result is

$$\lim_{b \to \infty} \int_1^b x e^{-x} dx = \frac{2}{3} - \lim_{b \to \infty} \frac{b+1}{e^b} = \frac{2}{3} - \lim_{b \to \infty} \frac{1}{e^b} = \frac{2}{3} - 0 = \frac{2}{3}$$

Therefore $\int_e^{\infty} \frac{\ln x}{x^2} dx = \frac{2}{3}$

Example 7. Find $\int_9^{\infty} \frac{x}{\sqrt{1+x^2}} dx$

Let $u = 1 + x^2$; $du = 2x$; $x = 9 \Rightarrow u = 82$

$$\begin{aligned}
\int_9^{\infty} \frac{x}{\sqrt{1+x^2}} dx &= \lim_{b \to \infty} \int_{82}^b \frac{du / 2}{\sqrt{u}} = \frac{1}{2} \lim_{b \to \infty} \int_{82}^b u^{-1/2} du = \frac{1}{2} \lim_{b \to \infty} \frac{u^{1/2}}{1/2} \Big|_{82}^b \\
&= \lim_{b \to \infty} \left(\sqrt{b} - \sqrt{82} \right) = \infty
\end{aligned}$$

Therefore the integral diverges.

Example 8. Find $\int_5^\infty \frac{x}{(1+x^2)^4}dx$

Let $u = 1 + x^2$; $du = 2xdx$; $x = 5 \Rightarrow u = 1 + 5^2 = 26$; $x = \infty \Rightarrow u = \infty$

$$\int_5^\infty \frac{x}{(1+x^2)^4}dx = \int_{26}^\infty \frac{du/2}{u^4} = \frac{1}{2}\lim_{b\to\infty}\int_{26}^b u^{-4}du = \frac{1}{2}\lim_{b\to\infty}(1/-3)u^{-3}\Big|_{26}^b$$

$$= -\frac{1}{6}\lim_{b\to\infty}\left(\frac{1}{b^3} - \frac{1}{26^3}\right) = \frac{1}{6(26)^3} = \frac{1}{105456}$$

Example 9. The work required to escape from the Earth's gravity is

$$W = \int_a^\infty F(r)dr$$

where $a = 6378\,km$ is the Earth's radius, $F(x) = -\frac{GMm}{r^2}$ is the force of gravity, M is the mass of the earth, m is the mass of the rocket plus payload, and G is Newton's constant of gravity. Therefore

$$W = \int_a^\infty F(x)dx = \int_a^\infty \frac{GMm}{r^2}dr = GMm\int_a^\infty r^{-2}dr = GMm(-1/r)\Big|_a^\infty$$

$$= -GMm\left(\frac{1}{\infty} - \frac{1}{a}\right) = \frac{GMm}{a}$$

Example 10. Show that if gravity was $1/r$ instead of $1/r^2$ it would be impossible to escape from the Earth's gravity.

$$W = \int_a^\infty F(x)dx = \int_a^\infty \frac{GMm}{r}dr = GMm\int_a^\infty r^{-1}dr = GMm(\ln r)\Big|_a^\infty$$

$$= GMm(\ln\infty - \ln a) = \infty$$

Example 11. Find $\int_1^\infty \frac{1}{x+x^2}dx$

Using partial fractions, we write

$$\frac{1}{x+x^2} = \frac{1}{x(1+x)} = \frac{A}{x} + \frac{B}{1+x} = \frac{A(1+x)+Bx}{x(1+x)}$$

Equating the numerators gives

$$1 = A(1+x) + Bx$$

Setting x=0 gives A=1

Setting x=-1 gives B=-1

Therefore

$$\int_1^\infty \frac{1}{x+x^2}dx = \int_1^\infty \left(\frac{A}{x} + \frac{B}{1+x}\right)dx = \int_1^\infty \left(\frac{1}{x} - \frac{1}{1+x}\right)dx$$

$$= \lim_{b\to\infty} \left(\ln x - \ln(1+x)\right)\Big|_1^b$$

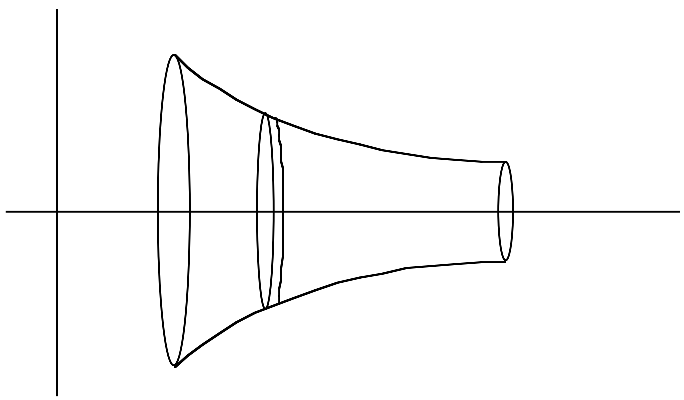
$$= \lim_{b\to\infty} \ln\frac{x}{1+x}\Big|_1^b$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{b}{1+b}\right) - \ln\left(\frac{1}{1+1}\right)\right]$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{2b}{1+b}\right)\right]$$

$$= \ln\left[\lim_{b\to\infty}\frac{2b}{b+1}\right] = \ln 2$$

Example 12. Find the volume of the surface of revolution obtained by rotating the curve $y = 1/x$, $1 \le x < \infty$ around the x axis.



The volume element (illustrated disc) is

$$2V = \pi r^2 dx = \pi (1/x)^2 dx$$

Therefore the volume is

$$V = \int_1^\infty \frac{\pi}{x^2}dx = \pi\int_1^\infty x^{-2}dx = \pi(-1/x)\Big|_1^\infty = \pi\left(-\frac{1}{\infty} - \frac{-1}{1}\right) = \pi$$

Example 13. Find the surface area of the surface of revolution in the previous example.

The area element is

$$dA = 2\pi r ds = 2\pi\left(\frac{1}{x}\right)\sqrt{1+(dy/dx)^2}\,dx$$

where $y = 1/x$. Hence $dy/dx = -1/x^2$ and therefore $(dy/dx)^2 = 1/x^4$. Therefore

$$dA = \frac{2\pi}{x}\sqrt{1+1/x^4}\,dx = \frac{2\pi}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx = \frac{2\pi}{x^3}\sqrt{1+x^4}\,dx$$

Therefore the total area is

$$A = \int_{x=1}^{x=\infty} dA = \int_1^\infty \frac{2\pi}{x^3}\sqrt{1+x^4}\,dx = 2\pi\int_1^\infty \frac{\sqrt{1+x^4}}{x^3}\,dx$$

Observe that

$$1+x^4 > x^4 \Rightarrow \sqrt{1+x^4} > x^4$$

$$\Rightarrow \frac{\sqrt{1+x^4}}{x^3} > \frac{x^4}{x^3} = x$$

Therefore

$$A = 2\pi\int_1^\infty \frac{\sqrt{1+x^4}}{x^3}\,dx > 2\pi\int_1^\infty x\,dx = 2\pi \lim_{b\to\infty} \left.\frac{x^2}{2}\right|_1^\infty = \infty$$

Thus the area integral diverges: the surface area is infinite!

The obvious conclusion from the last two examples is that if you turn this object on its side, **YOU CAN FILL IT UP BUT YOU CAN'T PAINT IT!**