6.5 Work

In physics, the work done in moving an object from $x=a$ to $x=b$ "against" a force $F(x)$ is

$$W = \sum Force \times distance = \int_a^b F(x)dx \qquad (1)$$

UNITS:

Force is measured in Newtons, where $1N = 1kg - m / sec^2$

Distance is measured in meters

Work is measured in Joules, where $1J = 1N - m = 1kg - m^2 / sec^2$

SPRINGS. The force exerted by a spring that is stretched out a distance x longer than its resting length is

$$F = -kx$$

where the minus sign indicates that the force is in the opposite direction of the extension of the spring (the book omits the minus sign). The units of the spring constant are Newtons/meter. Thus the force required to extend the spring is $-F=kx$.

The book does not deal with the sign of the force, and does not clearly distinguish beteen the force exerted by the spring and force required to extend the spring, and only talks about absolute value, i.e., it uses

$$|F| = kx$$

but does not explicitly deal with the minus sign or show the absolute value signs.

By equation (1) the work done in extending a spring from its natural length to a length L is

$$W = \int_0^L kxdx = \frac{1}{2}kx^2\Big|_0^L = \frac{1}{2}kL^2$$

Summary of spring formulas:

$F = kx$	force required to extend the spring a distance x
$W = \frac{1}{2}kx^2$	work it takes to extend the spring a distance x

Example 1. A force of 6 pounds keeps a spring stretched one half of a foot beyond its normal length. Find the spring constant.

$$|F| = |kx| \Rightarrow 6 \;\; pounds \;\; = \;\; k \times (0.5 \;\; foot)$$

$$\Rightarrow k = \frac{6}{0.5}\frac{pounds}{foot} = 12 pounds / foot$$

Example 2. How much work is done in stretching the spring in the previous example a total of two feet?

$$W = \frac{1}{2}kx^2 = \frac{1}{2}(12 \;\; pounds / foot) \times (2 feet)^2 = 24 foot - pounds$$

Example 3. If the natural length of a spring is 0.2 meters and it takes a force of 12 N to keep it extended by an additional 0.04 meters, find the work done in stretching the spring form its natural length to a length of 0.3 meters.

The first part of the problem gives you the information it takes to calculate the spring constant. A force for 12 N is needed to extend the spring by 0.04 meters from its resting length. The force of 12 N exactly balances the spring force, so $12 = -F = kx$ and therefore

$$12 = k(.04) \Rightarrow k = 12 / .04 = 300$$

Therefore the spring force is

$$F = -300x$$

Since the spring needs to be extended from a natural length of 0.2 meters to a total extension of 0.3 meters, the amount of the extension is 0.3 m – 0.2 m = 0.2 m.

Therefore the work done in stretching the spring from its resting length (x=0) to a total length of 0.3 meters is

$$W = \int_0^{.1} -300x\,dx = -300(x^2/2)\Big|_0^{.1} = -300(.01/2) = -1.5J$$

The negative sign indicates that work is done by the object, i.e., the net energy lost moving the spring is 1.5 J.

PUMPING WATER. The force working against us is gravity, and $F = -mg$ where m is the mass of an object and g is a constant. Since

$$density = mass / volume$$

then

$$mass = (volume) \times (density)$$

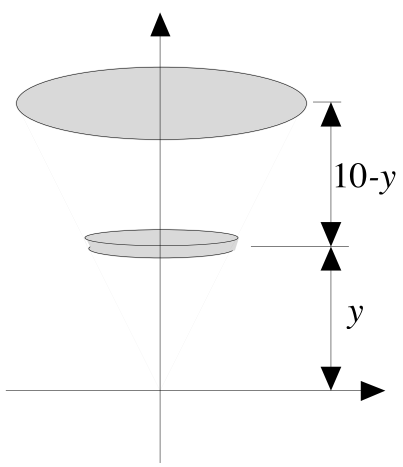
and therefore the force on a volume element dV of density δ is

$$F = -mg = -g\delta dV$$

NOTE: The book gives density in POUNDS/square foot, which is NOT REALLY DENSITY, it is (mass)(g)/square foot. So if density is given in POUNDS/square foot, then

$$F = -\delta dV$$

Example 4. Find the work required to remove the water from a conical tank of height 10 feet and radius 4 feet at the top, if the density of water is 62.4 pounds/cubic foot.



The cylindrical slab of water at y has a radius (4/10)y (because the line at edge of the cone goes through the origin and the point (4,10)). The volume of this slab is

$$dV = \pi r^2 dy = \pi\left(\frac{4}{10}y\right)^2 dy = \frac{16\pi}{100}y^2 dy$$

The work required to move this slab to the top of the cone is

$$dW = (force) \times (distance)$$
$$= \delta dV(10 - y) = \delta(10 - y)\frac{16\pi}{100}y^2 dy$$

Therefore the work to remove all of the water from the tank is

$$W = \int_{y=0}^{y=10} dW = \int_0^{10} \delta(10 - y)\frac{16\pi}{100} y^2 dy = \frac{16(62.4)}{100}\pi \int_0^{10} (10y^2 - y^3)$$

$$= 9.984\pi \left(\frac{10y^3}{3} - \frac{y^4}{4} \right)\Bigg|_0^{10} = 9.984\pi \left(\frac{10(10)^3}{3} - \frac{10^4}{4} \right)$$

$$= 9.984\pi \left(\frac{10{,}000}{3} - \frac{10{,}000}{4} \right) = 99{,}840(1/3 - 1/4)\pi$$

$$= \frac{99{,}840}{12}\pi = 8320\pi \approx 26{,}138$$