Math 150A – Exam 2 Answers
Section 15608

Your Name Here
2 October 2009

Rules

There are 5 problems. Complete all problems. Clearly indicate your your answer to each problem. Show all your work - partial credit may be given for some problems, but only if you show your work.
The exam is closed book and closed notes. Calculators are allowed. PDA's, computers, cell phones are prohibited.

Student Conduct Certification

This certification must be signed or your exam will not be graded.

I certify that I have read and understand the rules of the exam, and further, that the work shown in this examination is my own and that it has been completed in accord with the California State University Northridge student conduct code. I also understand that failure to abide by the student conduct code is subject to discipline as provided in sections 41301 through 41304 of Title 5, California Code of Regulations.

Sign here: ____________________

Problem	Points/Total
1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

Some Formulas

$$(uv)' = uv' + vu' \qquad \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x \qquad (\sec x)' = \sec x \tan x$$

$$(\csc x)' = -\csc x \cot x \qquad (\cot x)' = -\csc^2 x$$

$$x^{1/n} = \sqrt[n]{x} \qquad 1/x^n = x^{-n}$$

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m \qquad \sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos^2 x + \sin^2 = 1$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin 2x = 2 \sin x \cos x$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$

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1. Find each of the following derivatives and simplify:

   (a) $y = (x^4 - 3x^2 + 5)^3$. By the chain rule

$$\begin{aligned} y' &= 3(x^4 - 3x^2 + 5)^2 \frac{d}{dx}(x^4 - 3x^2 + 5) \\ &= 3(x^4 - 3x^2 + 5)^2 (4x^3 - 6x) \\ &= 6x(x^4 - 3x^2 + 5)^2 (2x^2 - 3) \end{aligned}$$

   (b) $y = \dfrac{t}{1 - t^2}$. Use the quotient rule:

$$y' = \frac{(1 - t^2)\frac{d}{dt}(t) - t\frac{d}{dt}(1 - t^2)}{(1 - t^2)^2} = \frac{1 - t^2 - t(-2t)}{(1 - t^2)^2} = \frac{1 + t^2}{(1 - t^2)^2}$$

   (c) If $f(x) = \sin(g(x))$, find $f'(x)$ in terms of $g'(x)$. Use the chain rule:

$$f'(x) = \frac{d}{dx}(\sin(g(x)) = \cos(g(x))\frac{d}{dx}g(x) = g'(x)\cos g(x)$$

   (d) If $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$, find $h'(x)$ in terms of $f'(x)$ and $g'(x)$. Use the quotient rule and then the product rule:

$$\begin{aligned} h'(x) &= \frac{(f(x) + g(x))\frac{d}{dx}(f(x)g(x)) - f(x)g(x)\frac{d}{dx}(f(x) + g(x))}{(f(x) + g(x))^2} \\ &= \frac{(f(x) + g(x))(f(x)g'(x) + f'(x)g(x)) - f(x)g(x)(f'(x) + g'(x))}{(f(x) + g(x))^2} \\ &= \frac{\begin{pmatrix} f(x)f(x)g'(x) + f(x)f'(x)g(x) + g(x)f(x)g'(x) + g(x)f'(x)g(x) \\ \hfill -f(x)g(x)f'(x) - f(x)g(x)g'(x) \end{pmatrix}}{(f(x) + g(x))^2} \\ &= \frac{(f(x))^2 g'(x) + (g(x))^2 f'(x)}{(f(x) + g(x))^2} \end{aligned}$$

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2. Find y' for each of the following:

   (a) $y = \cos(x^3 \tan x)$. By the chain rule, then the product rule:

$$\begin{aligned} y' &= -\sin(x^3 \tan x) \frac{d}{dx}(x^3 \tan x) \\ &= -\left(\sin(x^3 \tan x)\right)(3x^2 \tan x + x^3 \sec^2 x) \end{aligned}$$

   (b) $y = \left(x + \frac{1}{x^2}\right)^{\sqrt{7}}$. By the chain rule:

$$\begin{aligned} y' &= \sqrt{7}\left(x + \frac{1}{x^2}\right)^{\sqrt{7}-1} \frac{d}{dx}\left(x + \frac{1}{x^2}\right) \\ &= \sqrt{7}\left(x + \frac{1}{x^2}\right)^{\sqrt{7}-1} \left(1 - \frac{2}{x^3}\right) \end{aligned}$$

   (c) $y = \dfrac{1}{\sqrt[3]{x + \sqrt{x}}}$ rewrite as $y = (x + \sqrt{x})^{-1/3}$; then use chain rule:

$$\begin{aligned} y' &= -\frac{1}{3}(x + \sqrt{x})^{-4/3} \frac{d}{dx}(x + \sqrt{x}) \\ &= -\frac{1}{3}(x + \sqrt{x})^{-4/3}(1 + \frac{1}{2}x^{-1/2}) \\ &= -\frac{1 + \frac{1}{2\sqrt{x}}}{3\sqrt[3]{(x + \sqrt{x})^4}} \end{aligned}$$

   (d) $x \tan y = y - 1$. Use implicit differentiation:

$$\begin{aligned} \frac{d}{dx}(x \tan y) &= \frac{d}{dx}(y) \\ x(\tan y)' + (\tan y)(x)' &= y' && \text{product rule} \\ x((\sec^2 y)y') + \tan y &= y' && \text{chain rule on } (\tan y)' \\ y'(x \sec^2 y - 1) &= -\tan y && \text{algebra} \\ y' &= \frac{\tan y}{1 - x \sec^2 y} \end{aligned}$$

3. Solve the following problems using implicit differentiation.

   (a) Find the equation of the tangent line to $x^2 + 2xy - y^2 + x = 2$ at the point (1,2) <u>and simplify to the form $y = mx + b$</u>.
   Differentiating implicitly,

$$2x + 2(xy' + y) - 2yy' + 1 = 0$$

   Substitute $y' = m$, $x = 1$, $y = 2$, to get the slope:

$$2(1) + 2(1)(m) + 2(2) - 2(2)(m) + 1 = 0$$

$$2 + 2m + 4 - 4m + 1 = 0$$

$$7 - 2m = 0$$

$$m = 3.5$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 3.5(x - 1) = 3.5x - 3.5$$

$$y = 3.5x - 1.5 = \frac{7}{2}x - \frac{3}{2}$$

   (b) Find y'' for $x^4 + y^4 = 9$ by implicit differentiation <u>and simplify</u>. Differentiating implicitly:

$$4x^3 + 4y^3y' = 0 \implies y' = -\frac{4x^3}{4y^3} = -\frac{x^3}{y^3}$$

   Differentiate again, using the quotient rule:

$$\begin{aligned} y'' &= -\frac{(y^3)(x^3)' - (x^3)(y^3)'}{y^6} \\ &= -\frac{3x^2y^3 - 3x^3y^2y'}{y^6} \\ &= -\frac{3x^2y^3 - 3x^3y^2\left(\frac{-x^3}{y^3}\right)}{y^6} \times \frac{y^3}{y^3} \\ &= -\frac{3x^2y^6 + 3x^6y^2}{y^9} \\ &= -3x^2y^2\left(\frac{y^4 + x^4}{y^9}\right) \\ &= \frac{-3x^2}{y^7}(x^4 + y^4) = \frac{-3x^2}{y^7}(9) = -27\frac{x^2}{y^7} \end{aligned}$$

4. Find the following limits.

(a) $\lim_{t\to 0} \frac{\tan 7t}{\sin 3t}$

$$\begin{aligned}\frac{\tan 7t}{\sin 3t} &= \frac{\sin 7t}{1} \times \frac{1}{\cos 7t} \times \frac{1}{\sin 3t} = 7t \times \frac{\sin 7t}{7t} \times \frac{1}{\cos t} \times \frac{1}{3t} \times \frac{3t}{\sin 3t} \\ &= \frac{7t}{3t} \times \frac{\sin 7t}{7t} \times \frac{1}{\cos t} \times \frac{3t}{\sin 3t} = \frac{7}{3} \times \frac{\sin 7t}{7t} \times \frac{1}{\cos t} \times \frac{3t}{\sin 3t} \\ &\to \left(\frac{7}{3}\right)(1)(1)(1) = \frac{7}{3}\end{aligned}$$

(b) $\lim_{t\to 0} \frac{\sin^2(3t)}{t^2}$

$$\frac{\sin^2(3t)}{t^2} = \left(\frac{\sin 3t}{t}\right)\left(\frac{\sin 3t}{t}\right) = 9\left(\frac{\sin 3t}{3t}\right)\left(\frac{\sin 3t}{3t}\right) = (9)(1)(1) = 9$$

(c) $\lim_{x\to 0} \frac{\sin x}{x + \tan x}$

$$\frac{\sin x}{x + \tan x} = \frac{\frac{\sin x}{x}}{\frac{x}{x} + \left(\frac{\sin x}{x} \times \frac{1}{\cos x}\right)} \to \frac{1}{1 + (1)(1)} = \frac{1}{2}$$

(d) $\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \sin\frac{\pi}{6}}{h}$

<u>Method 1:</u>

$$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \sin\frac{\pi}{6}}{h} = f'\left(\frac{\pi}{6}\right) \text{ where } f(x) = \sin x$$

Since $f'(x) = \cos x$, the limit is $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

<u>Method 2 (Direct Calculation):</u>

$$\begin{aligned}\frac{\sin\left(\frac{\pi}{6} + h\right) - \sin\frac{\pi}{6}}{h} &= \frac{\sin\frac{\pi}{6}\cos h + \cos\frac{\pi}{6}\sin h - \sin\frac{\pi}{6}}{h} \\ &= \frac{\sin\frac{\pi}{6}\,(\cos h - 1)) + \cos\frac{\pi}{6}\sin h}{h} \\ &= \sin\frac{\pi}{6} \times \frac{\cos h - 1}{h} + \cos\frac{\pi}{6} \times \frac{\sin h}{h} \\ &\to \frac{1}{2} \times 0 + \frac{\sqrt{3}}{2} \times 1 = \frac{\sqrt{3}}{2}\end{aligned}$$

5. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $h(x) = g(f(x))$, find $h'(1)$

$$\begin{aligned} h'(x) &= g'(f(x))f'(x) \\ h'(1) &= g'(f(1))f'(1) = g'(3)(4) = (9)(4) = 36 \end{aligned}$$

(b) If $h(x) = f(f(x))$, find $h'(2)$

$$\begin{aligned} h'(x) &= f'(f(x))f'(x) \\ h'(2) &= f'(f(2))f'(2) = f'(1)(5) = (4)(5) = 20 \end{aligned}$$

(c) If $h(x) = g(g(x))$, find $h'(3)$

$$\begin{aligned} h'(x) &= g'(g(x))g'(x) \\ h'(3) &= g'(g(3))g'(3) = g'(2)(9) = (7)(9) = 63 \end{aligned}$$

(d) Instead of using the table, suppose that $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$. Then find $F'(1)$ if $F(x) = f(xf(xf(x)))$.

$$\begin{aligned} F'(x) &= f'(xf(xf(x))\frac{d}{dx}\left(xf(xf(x))\right) \\ &= f'(xf(xf(x)) \times \left(x\frac{d}{dx}f(xf(x)) + f(xf(x))\frac{d}{dx}x\right) \\ &= f'(xf(xf(x)) \times \left[xf'(xf(x))\frac{d}{dx}(xf(x)) + f(xf(x))\right] \\ &= f'(xf(xf(x)) \times [xf'(xf(x)) \times (xf'(x) + f(x)) + f(xf(x))] \\ F'(1) &= f'(1 \cdot f(1 \cdot f(1))) \times [1 \cdot f'(1 \cdot f(1)) \cdot (1 \cdot f'(1) + f(1)) + f(1 \cdot f(1)))] \\ &= f'(f(f(1))) \times [f'(f(1))(4+2)) + f(f(1)))] \\ &= f'(f(2)) \times [6f'(2) + f(2))] \\ &= f'(3) \times (6 \times 5 + 3) \\ &= 6 \times 33 = 198 \end{aligned}$$