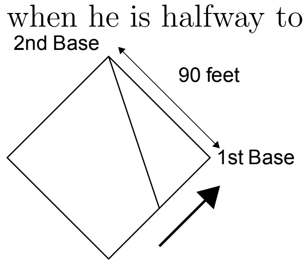
Worksheet 5 - Solutions (Corrected 10/28/09)

Make sure to specify the correct units on all solutions.

1. Suppose a ball is throuwn vertically upward with a speed of 80 feet/second and that its height after t seconds is given by $h(t) = 80t - 16t^2$. Find (a) the maximum height of the ball; (b) the speed when the ball is going up at a height of 96 feet; (c) the speed when the ball is going down at a height of 96 feet; (d) the speed when the ball hits the ground.

$$h'(t) = 80 - 32t \qquad h'(t) = 0 \implies 0 = 80 - 32t \implies t = 80/32 = 2.5 \text{ sec}$$
$$h(2.5) = 80(2.5) - 16(2.5^2) = 100 \text{ feet}$$
$$h(t) = 96 \implies 96 = 80t - 16t^2 \implies 16t^2 - 80t + 96 = 0 \implies t^2 - 5t + 6 = 0$$
$$\implies 0 = (t-2)(t-3)$$
$$t = 2 \text{ (going up)}: h'(2) = 80 - 32(2) = 16 \text{ feet/sec}$$
$$t = 3 \text{ (going down)}: h'(3) = 80 - 32(3) = -16 \text{ feet/sec}$$

2. A baseball diamond is a square with sides 90 feet. A batter hits the ball and runs toward first base with a speed of 24 feet/second. At what rate is his distance from second base decreasing when he is halfway to first base?



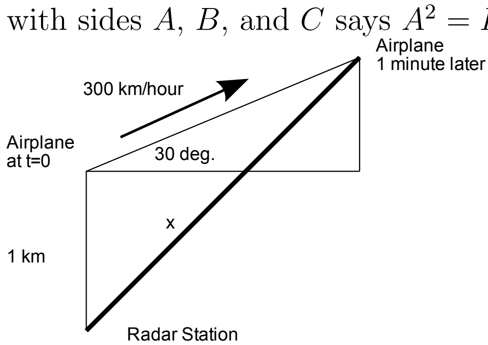
Let D = distance from player to 2nd base and x the distance to 1st base. Then by the Pythagorean theorem,

$$D^2 = x^2 + 90^2 \implies 2DD' = 2xx' \implies D' = \frac{x}{D}x'$$

When $x = 45$, $D = \sqrt{45^2 + 90^2} = \sqrt{2025 + 8100} = \sqrt{10125} = 45\sqrt{5}$ feet hence

$$D' = \frac{45}{45\sqrt{5}}(24) = \frac{24}{\sqrt{5}} \approx 10.73 \text{ feet/sec}$$

3. A plane flying with a constant speed of 300 km/hour passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30 degrees. At what rate is the distance from the plane to the radar station increasing one minute later. Hint: The law of cosines for a triangle with sides A, B, and C says $A^2 = B^2 + C^2 - 2BC\cos\theta$ where θ is the angle opposite side A.



Let x be the distance from radar station to the plane, and D the distance of the plane from the point 1 km above the radar station. By the law of cosines,

$$x^2 = D^2 + 1^2 - (2)(1)(D)\cos(30 + 90)$$
$$= D^2 + 1 - 2(D)(-1/2) \text{ because } \cos 120 = \frac{1}{2}$$
$$= D^2 + 1 + D$$
$$2xx' = 2DD' + D' = (2D+1)D' \implies x' = \frac{2D+1}{2x}D'$$

After 1 minute, the plane has flown 300/60=5 km, hence $D = 5$. Thus

$$x^2 = 5^2 + 1 + 5 = 31 \implies x = \sqrt{31}$$

so that

$$x' = \frac{2D+1}{2x}D' = \frac{2(5)+1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296.35$$

4. To resistors connected in parallel have an effective total resistance R where $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If resistors R_1 and R_2 are increasing at rates of 0.3 Ohms/second and 0.2 Ohms/second, resepctively, how fast is R changing when $R_1 = 80$ Ohms and $R_2 = 100$ Ohms? (The Ohm is the unit of resistance).

   First, get the total resistance,

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{100} = \frac{180}{8000} = \frac{9}{400} \implies R = \frac{400}{9}$$

   Differentiating implicitly,

$$\begin{aligned} -\frac{1}{R^2}\frac{dR}{dt} &= -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} \\ \implies \frac{dR}{dt} &= \frac{R^2}{R_1^2}\frac{dR_1}{dt} + \frac{R^2}{R_2^2}\frac{dR_2}{dt} = \left(\frac{R}{R_1}\right)^2 \frac{dR_1}{dt} + \left(\frac{R}{R_2}\right)^2 \frac{dR_2}{dt} \\ &= \left(\frac{400}{9(80)}\right)^2 (.3) + \left(\frac{400}{9(100)}\right)^2 (.2) \approx .132 \text{ Ohms/sec} \end{aligned}$$

5. Find a linear approximation to $f(x) = \frac{1}{\sqrt{2+x}}$ near $x = 0$.

$$f(0) = \frac{1}{\sqrt{2}} \implies f'(x) = -\frac{1}{2}(2+x)^{-3/2} \implies f'(0) = -(2^{-3/2})/2 = -2^{-5/2}$$

   Hence the linearization is

$$f(x) \approx f(0) + (x-0)f'(0) = \frac{1}{\sqrt{2}} - 2^{-5/2}x$$

6. Use differentials to approximate $\sqrt{99.8}$.

   Since $99.8 = 100 - 0.2$ then $\sqrt{99.8}$ is near $\sqrt{100} = 10$. Let $f(x) = \sqrt{x}$. Then $\sqrt{99.8} = f(99.8)$. Since $f'(x) = -\frac{1}{2\sqrt{x}} \implies f'(100) = -\frac{1}{2\sqrt{100}} = -\frac{1}{20}$ the linearization is

$$f(x) \approx f(100) + (x-100)f'(100) = \sqrt{100} + \frac{x-100}{20} = 10 + \frac{x-100}{20}$$

   Therefore $\sqrt{99.8} \approx 10 + \frac{99.8-100}{20} = 10 - \frac{.2}{20} = 9.99$

7. Use differentials to estimate the amout of paint (in liters) to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 meters (1 meter = 100 cm, 1 liter = $10 \times 10 \times 10 = 1000$ cm^3).(A hemisphere is half of a sphere.)

   The volume of a sphere is $V = \frac{4}{3}\pi r^3$ hence $V = \frac{2}{3}\pi r^3$ for a hemisphere. Differentiating,

$$dV = 2\pi r^2 dr = 2\pi(2500 \text{ cm})^2(.05 \text{ cm}) = 625{,}000\pi \text{ cm}^3 = 625\pi \text{ liter}$$