Math 103 Review

Fall 2008

Lines (1.1, 1.2)

1. **General form**: $Ax + By = C$ where A, B, and C are any three numbers.
2. **Slope**=$\dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are any two points on the line.
3. **Slope-intercept form:** $y = mx + b$ where $m =$ the slope and $b =$ the y-intercept.
4. **Point-slope form**: $y - y_1 = m(x - x_1)$ where $m =$ the slope and (x_1, y_1) is any point on the line.
5. **Slope-slope form**: $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where $a = x-$intercept and $b = y-$intercept.
6. **Vertical** line: $x =$constant (slope=∞ or undefined)
7. **Horizontal** line: $y =$constant (slope $= 0$)
8. **x-intercept**: where the line crosses the $x-$axis; solve for x when $y = 0$.
9. **y-intercept**: where the line crosses the $y-$axis; solve for y when $x = 0$

Functions (2.1, 2.2)

1. A **function** is a correspondence between two sets of elements such that to each element in the first set (the **input** or **domain**) there corresponds precisely one element in the second set (the **output** or **range**).
2. **Domain**: the **input** (usually the x-axis). The set of x-values for which the equation that defines the function makes sense (no dividing by zero, do roots of negatives).
3. **Range**: the **output** (usually the y-axis).
4. **Vertical line test**: If any vertical line passes through more than one point in the graph, it is NOT a function. If all vertical lines pass through at most one point, then it is a function.
5. **Vertical Shift Up**: $f(x) + k$
6. **Vertical Shift Down**: $f(x) - k$
7. **Horizontal Shift Left**: $f(x + h)$
8. **Horizontal Shift Right**: $f(x - h)$
9. **Vertical Stretch**: $Af(x), A > 1$
10. **Vertical Shrink**: $Af(x), A < 1$
11. **Reflection**: $-f(x)$

Parabolas and polynomials (2.3)

1. **General form**: $y = ax^2 + bx + c$
   - If $a > 0$, opens upwards
   - If $a < 0$, opens downwards
2. **Vertex form**: $y = a(x - h)^2 + k$, where the vertex is at (h, k)
3. To convert from general form to vertex form, **complete the squares**

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \end{aligned}$$

4. To find the $x-$ intercepts, solve for x when $y = 0$.
5. If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
6. The vertex of a parabola occurs when $x = -b/2a$ (solve for x in $f'(x) = 0$.
7. **Polynomial**: $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$
   - Degree = largest exponent
   - Turning points are when $f'(x) = 0$
   - Num. Turning points $\leq$ degree $- 1$
   - Degree $\geq$ Num. Turning Points $+ 1$
   - Even degree $\iff$ Odd numb. turning points
   - Odd degree $\iff$ Even numb. turning points

Rational Functions (2.3, 10.3)

1. **Rational Function**: $\dfrac{f(x)}{g(x)}$ where f and g are polynomials.
2. **Asymptotes** of $\dfrac{ax + b}{cx + d}$:
   - **Vertical**: Solve $cx + d = 0$ for x
   - **Horizontal**: $y = a/c$

Exponentials and Logarithms (2.4, 2.5)

$$\text{If } y = e^x \text{ then } x = \ln y$$

$$\ln(ab) = \ln a + \ln b$$

$$\ln \frac{a}{b} = \ln a - \ln b$$

$$\ln(e^x) = x, \quad e^{\ln x} = x$$

$$\ln e = 1, \quad \ln 1 = 0$$

$$\ln x^n = n \ln x$$

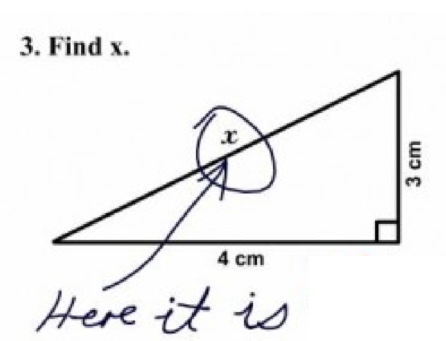
Simple & Compound Interest (3.1, 3.2, 11.1)

1. **Simple Interest**: $A = P(1 + rt)$
2. **Compound Interest**: $A = P\left(1 + \frac{r}{m}\right)^{mt}$
3. **Continuous Compound Interest**: $A = Pe^{rt}$

Limits and Continuity (10.1, 10.2, 10.3)

1. **Limit from Left**: $\lim_{x \to a^-} f(x) = L$
2. **Limit from Right**: $\lim_{x \to a^+} f(x) = L$
3. **Limit**: $\lim_{x \to a^+} f(x) = L$ if and only if
$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$
4. **Properties of Limits**:
$$\lim_{x \to a} C = C, \quad \lim_{x \to a} x = a$$
$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$
$$\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$$
$$\lim_{x \to a} [f(x)/g(x)] = \left(\lim_{x \to a} f(x)\right) / \left(\lim_{x \to a} g(x)\right)$$
$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)}$$
$$\lim_{x \to \infty} \frac{ax + b}{cx + d} = \frac{a}{c}$$
5. A function $f(x)$ is **continuous at x=a** if and only if $\lim_{x \to a} f(x) = f(a)$
6. Continuous $\iff$ can draw graph without lifting pencil
7. Discontinuous at jumps and holes

From a Real Exam:



Derivatives (10.4, 10.5, 11.3, 11.4)

1. **Difference Quotient**: $\frac{f(x+h) - f(x)}{h}$
2. **Definition of a Derivative**:
$$y' = f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
3. **Power Law**: $\frac{d}{dx} x^n = nx^{n-1}$
4. **Sum/Difference**: $(f(x) + g(x))' = f'(x) + g'(x)$
5. **Constant times a function**:
$$(cf(x))' = \frac{d}{dx}(cf(x)) = c\frac{df(x)}{dx} = cf'(x)$$
6. **Product Rule**: $(LR)' = LR' + RL'$
7. **Quotient Rule**: $\left(\frac{T}{B}\right)' = \frac{BT' - TB'}{B^2}$
8. **Chain Rule**: $[u(x)^n]' = n\,[u(x)]^{n-1}\, u'(x)$
9. The value of the derivative gives the **slope of the tangent line**, e.g., $f'(3)$ gives the slope of the tangent line of $f(x)$ at $x = 3$.
10. The slope (derivative) is zero when the tangent is horizontal.
11. The slope (derivative) is zero at a maximum or minimum (unless the maximum or minimum occurs at an endpoint).
12. **Absolute maximum or minimum**:
    (a) Calculate $f'(x)$
    (b) Solve $f'(x) = 0$ for x to get the turning points.
    (c) Plug x back into $f(x)$ to find the y at each turning point.
    (d) If the domain is a finite interval $[a, b]$ calculate $f(a)$ and $f(b)$
    (e) The absolute maximum (minimum) corresponds to the point in (12c) or (12d) with the largest (smallest) y value.
13. The slope of the **secant line** through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is
$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
and the equation of the line through those points is
$$y - f(x_1) = m(x - x_1)$$
14. **The equation of the tangent line** to $f(x)$ at $x = a$ is
$$y - y_1 = m(x - x_1)$$
where $x_1 = a$, $y_1 = f(a)$ and $m = f'(a)$.

Price-Demand (2.1, 10.7)

1. **Price-demand equation**: an equation that relates p (price per item) to x (demand). Typically the price-demand equation looks something like $p = a + bx$.

2. Suppose $ax + bp = c$ where x =demand and p = price. Then

   - To get **demand as a function** $f(p)$ **of price**, solve the equation for x. Then $f(p)$ equals your formula for x which only depends on p and not x
   - to get **price as a function** $f(x)$ **of demand**, solve the equation for p. Then $f(x)$ equals your formula for p which only depends on x and not p.

Cost, Revenue, Profit (10.7, 10.11)

1. **Revenue**=(Price)×(Demand): $R(x) = xp(x)$ where p is the price per item.

2. **Cost** =(fixed cost) + (variable cost per item) × (number of items). Typically the Cost Function $C(x)$ looks like $C(x) = f + vx$ where f is the fixed cost and v is the variable cost per item.

3. **Profit** =(Revenue)−(Cost): $P(x) = R(x) - C(x)$

4. **Exact Marginal Cost** (**Revenue**, **Profit**) is the Cost (or Revenue or Profit) of producing **one additional item**, e.g.,

   - Exact Marginal Cost: $C(x + 1) - C(x)$
   - Exact Marginal Revenue: $R(x + 1) - R(x)$
   - Exact Marginal Profit: $P(x + 1) - P(x)$

5. **Marginal Cost** (or Revenue or Profit) is given by the Derivative:

   - Marginal Cost: $C'(x)$
   - Marginal Revenue: $R'(x)$
   - Marginal Profit: $P'(x)$

6. **Break Even Points**: $P(x) = 0$, i.e, $R(x) = C(x)$

7. **Maximum Revenue** occurs when $R'(x) = 0$. To find the demand that gives maximum revenue, solve $R'(x) = 0$ for x. To find the maximum revenue, substitute the x you found back into the revenue function $R(x)$.

8. **Maximum Profit** occurs when $P'(x) = 0$, i.e., when

   $$R'(x) = C'(x)$$

   To find the demand that gives maximum profit, solve $R'(x) = C'(x)$ for x. To find the value of the maximum profit, substitute the x your found into the equation for profit, $P(x) = R(x) - C(x)$.

Elasticity of Demand (11.7)

Definition:

$$E(p) = \frac{-pf'(p)}{f(p)}$$

where $x = f(p)$ is demand as a function of price.

1. Inelastic: $E(p) < 1$: demand is not very sensitive to price - a change in price only leads to a relatively small change in demand.

2. Elastic: $E(p) > 1$: demand is sensitive to price - a small change in price induces a large change in demand.

3. Unit (Unitary): $E(p) = 1$: change in demand is same as change in price (proportianally or percentage).

Systems of Equations (4.1, 4.2, 4.3)

$$ax + by = p$$

$$cx + dy = q$$

1. **Solution by addition**: any multiple of one equation may be added to (or subtracted from) another.

2. **Solution by substitution**.

   - Solve either equation for x
   - Subsitute the formula you find for x into the second equation
   - Solve this new equation for y
   - Substitute the value for y back into your first equation for x

3. **Solution by Matrices**. Simplify the augmented matrix

   $$\begin{pmatrix} a & b & p \\ c & d & q \end{pmatrix}$$

   to

   $$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \end{pmatrix}$$

   by a sequence of row operations:

   - Replace any row with a multiple of that row
   - Replace any row with the sum/difference of two rows
   - Switch any two rows

   Infinite number of solutions:

   $$\begin{pmatrix} 1 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$$

   No solution:

   $$\begin{pmatrix} 1 & 0 & x \\ 0 & 0 & y \end{pmatrix}$$

Matrix Operations (4.4, 4.5)

1. a_{ij} is the number in row i and column j
2. Matrix size is measured in rows $\times$ columns
3. Addition/Subtraction

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$$

4. Let A be a $m \times n$ matrix
Let B by a $p \times q$ matrix
Then AB is only defined if $n = p$ and then AB is a $m \times q$ matrix.
5. Matrix multiplication is defined by the row $\times$ column products:

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = aP + bQ + cR$$

6. the ij element of the product AB is given by

$$(\text{row } i \text{ of } A) \begin{pmatrix} \text{colum} \\ j \\ \text{of B} \end{pmatrix}$$

7. Some Matrix Products:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

8. Matrix Inverse: the Matrix A^{-1} is the matrix such that

$$A^{-1}A = AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

9. To find the inverse of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

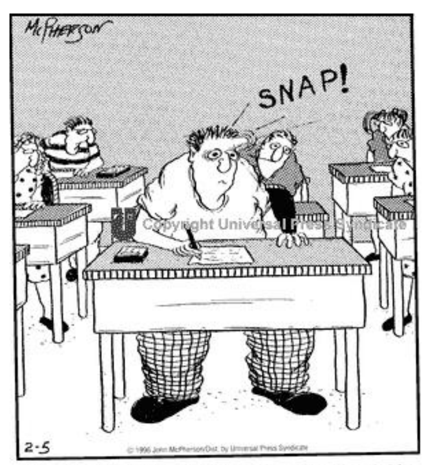
simplify the augmented matrix

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 0 & p & q \\ 0 & 1 & r & s \end{pmatrix}$$

"Just a darn minute! — Yesterday you said that X equals two!"



Unfortunately, Brad had neglected to stretch his brain before taking the big algebra midterm.